

The Interpretations and Applications of Boethius's *Introduction to the Arithmetic* II, 1 at the End of the 10th Century

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Abstract:

This paper deals with comments and glosses to the first chapter of the second book of Boethius's *Introduction to Arithmetic* from the last quarter of the 10th century. Those texts were written by Gerbert of Aurillac (*Scholium ad Boethii Arithmeticae Institutionem* l. II, c. 1), Abbo of Fleury (commentary on the *Calculus* by Victorius of Aquitaine, the so-called *De numero, mensura et pondere*), Notker of Liège (*De superparticularibus*) and by the anonymous author (*De arithmetica Boetii*). The main aim of this paper is to show that Boethius's statements about the converting numerical sequences to equality from this work could be interpreted minimally in two different ways. This paper discussed also the application of this topic in other liberal arts (like astronomy, music, grammar etc.) and in playing game called *rithmomachia*, the medieval philosophers' game.

Keywords: Boethius, arithmetic, Gerbert of Aurillac, Abbo of Fleury, Notker of Liège

1. Preface

Since at least the time of the Pythagoreans, it has been a well-known fact in philosophical debates that mathematics and numbers are not only a tool to capture the quantitative aspects of reality, but that mathematical disciplines – such as those

that would later be known as the quadrivium, i.e., arithmetic, geometry, astronomy and music (Boethius [1999], p. 11) – are also essential parts of the ontological-metaphysical, theological, cosmological, epistemological-anthropological, and other related fields of research. The Divine Unit – the precise and perfect order determining natural events or the music of the spheres – represents just a brief selection of topics that make an art out of the science of numbers, an art linked to the very essence of the universe.

In the emerging medieval Latin Europe prior to 1000, the philosophical and theological dimensions of the mathematical arts – especially arithmetic as the first mathematical discipline (Boethius [1999], p. 12; Isidorus [1911], III, 1, 1; Cassiodorus [1937], p. 132) – were authoritatively backed by, for example, Aurelius Augustine, who in many parts of his work emphasized the importance of mathematics (and separately also of arithmetic) as a valuable exegetical tool (for example Augustinus [1962], II, 16, 25), as a way to God or as the very expression of God's wisdom – the order of numbers is marked by Augustine as consubstantial with God himself (Augustinus [1970], II, 11, 32). The basic textbook of the first quadrivium science was represented by Boethius's *Introduction to Arithmetic*, a Latin adaptation (respectively a free translation) of a Greek document of the same name by Nicomachus of Gerasa, a Neo-Pythagorean mathematician.

This study will focus on the four surviving texts from the late 10th century, which, directly or indirectly, respond to stimuli arising from Boethius's *Introduction to Arithmetic*, especially the interpretations, notes or commentaries to the first chapter of the second book of Boethius's work in which the question of ratios, numerical sequences and transfers of different ratios to an initial equality are solved. The aim of this paper is to show how the text by the "Last Roman" was worked with in the years prior to 1000. An essential part of this presentation is also a brief introduction to the selected topics of Nicomachean Neo-Pythagorean arithmetics, as mediated to the Latin-speaking world primarily by Boethius, and, finally, an effort to discover and briefly introduce possible reasons for the interest of early medieval thinkers in numerical ratios and sequences.

2. Comments and Glosses to *Introduction to Arithmetic* II, 1 from the late 10th century

Perhaps the most famous author of a coherent text from the late 10th century, focusing on the interpretation of Boethius's *Introduction to Arithmetic* II, 1, was Gerbert of Aurillac (before 950–1003), abbot of Bobbio (981/982–983/998), unconfirmed archbishop of Rheims (990/991–995/997), archbishop of Ravenna (998–999) and under the name of Sylvester II, Pope (999–1003).¹ The origins of Gerbert's scholarly reputation can be dated to the 70s of 10th century, when he began to cooperate with the Ottonian imperial dynasty and became a teacher in Rheims. Gerbert's contemporaries marvelled especially at his knowledge of the quadrivium and his unusual emphasis on the practical use of individual pieces of knowledge: the simplification of geometric and arithmetical calculations by means of a counting frame or abacus, the use of observational astronomical instruments, the construction of clocking mechanisms and his innovations for the organ of that time and also related instrumental techniques, etc. He seems to have mastered an approach to the liberal arts – especially during his approximately three-year stay (967–970) in the Iberian Peninsula – unusual for the given period. Among Gerbert's pupils or collaborators during his teaching in Reims was a monk, Constantine of Fleury, who at the end of the 970s – or early 80s – devoted several letters and essays to work on the abacus (Bubnov [1899], pp. 1–24), the construction of an observational hemisphere (Bubnov [1899], pp. 24–8), the interpretation of certain passages of Boethius's *Introduction to music* (Bubnov [1899], pp. 28–31) and probably also explanatory notes to Boethius's *Introduction to Arithmetic* II, 1, the so-called *Scholium ad Boethii Arithmetice Institutionem* I. II, c. 1 (Bubnov [1899], pp. 31–5), which was most likely written between 978 and 980, and in which he attempts to clarify a

¹ Information referring to his time and concerning his life and works is provided, above all, by Richer of Reims (Richerus Remensis [2000], p. 191–205). Substantial data also provides his correspondence (Gerbert [1966]), more detailed information about his life, works, resp. legend, see e.g. Brown [2010]; Flusche [2005]; Riché [1987]; Tosi [1985]; Oldoni [1977], [1980], [1983]; Darlington [1947] and many other sources.

process, not fully explained by Boethius, during which unequal ratios between numbers are converted to equality and identity.

Probably only slightly later is a mathematical treatise by Abbo of Fleury (ca. 945–1004), a monk, Prior and then Abbot (988–1004) of the monastery of St. Benedict in Fleury,² who after his studies (e.g. Paris or Rheims) in the 980s worked in a convent school in Ramsey as an English teacher. Probably in the first half of the 980s (Abbo [2003], p. xiv) he commented on *Calculus* by Victorius of Aquitaine, which he called *De numero, mensura et pondere* (Abbo [1982], p. 275, [2003], p. 65). This commentary on Victorius's preface to subsequent tables is therefore directly linked to Boethius's *Introduction to Arithmetic*, whose purpose and goal is directed elsewhere. Nevertheless, Abbo refers to Boethius's authoritative text several times (Abbo [2003], p. 87) and, above all, discusses in more detail the issues which the aforementioned introductory chapter of the second book of *Introduction to Arithmetic* deals with.

Although Gerbert writes his *Scholium* as an interpretation of Boethius, while Abbo's text is rather a commentary on Victorius, they both allude to the fact that the transfer of ratios expressing the inequality of numbers needs to be executed properly, avoiding confusion (confuse – see Bubnov [1899], p. 33; Abbo [2003], p. 86). It seems that they both respond to a certain unsatisfactory method of creating transitions between numerical sequences that was probably practised at the time. However, neither of the authors is explicit on this point, so it is only possible to speculate. From approximately the same period, another brief description of how to convert each numerical sequence to an original equality has survived. This text is called *De superparticularibus* (Bubnov [1899], pp. 297–9; see also Caiazza [2000], p. 118, 125; or Evans [1980], p. 264–6) and its authorship is attributed to Notker of Liège (ca. 940–1008), who apparently worked and studied at the Benedictine monastery of St. Gall and, after a stay at the Ottonian imperial

² The author of the biographical treatise about this persons from the early Middle Ages is Aimon of Fleury (Aimonius [1853]), even Abbo's correspondence is preserved and biographical data can be gained from some of his works (see Abbo [1853a] or [1853b]). For further information see also e.g. Dachowski [2008]; Riché [2004]; Engelen [1993]; Mostert [1987] and many others.

court, with the help of Otto I, became a bishop – from 972 – and prince-bishop – from 980 – in Liège, where he contributed significantly to the overall development of the city, including considerable and successful efforts to make a local school flourish.³ In the second half of the 990s he – like Gerbert – moved in the close circles of Emperor Otto III. There is no doubt that Abbo and Gerbert were familiar with the interpretation of transitions between sequences – usually associated with Notker –, which shows clear features of the confusion mentioned by them. Thus, between all three texts there may be factual connections suggested in the extant notes from the 12th century (Evans [1980], p. 267).

Notker's works have been preserved in a manuscript from the monastery in Tegernsee (now Munich, Bayerische Staatsbibliothek, CLM 18764, f. 78v–79r) as a postscript to Boethius's *Introduction to Arithmetic*. Part of this manuscript is also a preface (*accessus*) and commentary on Boethius's manuscript called *De aritmetica Boetii* edited by I. Caiazzo (Caiazzo [2000], pp 126–49; see also Evans [1978], p. 23). These texts most likely appeared in the late 10th century (Caiazzo [2000], pp 123–25), therefore creating an authentic complement to the three previously mentioned treatises.

3. Equality and inequality

A subject of interest for early medieval authors in the late 10th century was the main introductory passage of the second book of Boethius's *Arithmetics*, in which the author claims that any inequality derives from a former equality, analogous to the concept that everything material is composed of four elements (*elementa*), words are composed of letters (*litterae*) and music is based on sounds (*sonus*). All these components have originated from a former identity, equality and unity (Boethius [1999], pp. 93–4; Nicomachus [1866], pp. 73–74). And because everything is created according to numerical ratios (Boethius [1999], p. 14; Nicomachus [1866], p. 12) and

³ See especially the medieval biography *Vita Notgeri episcopi Leodiensis* (Kurth [1905b], pp. 10–5). Reports referring to that time see particularly *Anselmi gesta episcoporum Tungrensium, Traiectensium et Leodiensium* (Anselmus [1846], pp. 203–6). Furthermore see e.g. Delville et al. [2008]; G. Kurth [1905a] etc.

God Himself is originally one and the same undivided being (Boethius [1978], p. 28 or pp. 40–48), it is clear that any proportion or ratio has its roots in unity and equality (Boethius [1999], p. 80; Nicomachus [1866], p. 65; cf. also Abbo [2003], pp. 85–86).

Equality and inequality in *Arithmetic* are defined as two types of relative properties of numbers, i.e. such properties of numbers in which the relationship between the numerical value is considered (Boethius [1999], p. 55; Nicomachus [1866], p. 44). In the case of equality, the compared values are not smaller or larger, but identical, i.e. they are the same size, as if we were comparing ten and ten, three and three, cubit and cubit, foot and foot, etc. (Boethius [1999], p. 55; Nicomachus [1866], p. 44; see also Isidorus [1911], III, 6, 3; Cassiodorus [1937], p. 136; Caiazzo [2000], p. 131). The inequality of numbers is given by the fact that when comparing numerical values, one number is greater or smaller than the other – by this the two basic types of inequality are also determined. Larger or smaller inequality is then further divided into five basic types of inequality (Boethius [1999], p. 56; Nicomachus [1866], pp. 45–6; see also Isidorus [1911], III, 6, 1–2, 4–5; Cassiodorus [1937], p. 135), i.e. multiples and divisors (Boethius [1999], pp. 56–9; Nicomachus [1866], pp. 46–8; see also Isidorus [1911], III, 6, 5–6; Cassiodorus [1937], p. 136; Martianus [1983], p. 279; Abbo [2003], pp. 116–7), superparticular, subsuperparticular (Boethius [1999], pp. 60–3; Nicomachus [1866], pp. 49–50; see also Isidorus [1911], III, 6, 10; Cassiodorus [1937], pp. 136–7; Martianus [1983], p. 279; Abbo [2003], pp. 117–8), superpartient and subsuperpartient ratios (Boethius [1999], pp. 70–3; Nicomachus [1866], pp. 55–9; see also Isidorus [1911], III, 6, 8–9; Cassiodorus [1937], p. 137; Martianus [1983], p. 280; Abbo [2003], pp. 118–9), superparticular subsuperparticular multiples and divisors (Boethius [1999], pp. 73–8; Nicomachus [1866], pp. 59–63; see also Isidorus [1911], III, 6, 11–12; Cassiodorus [1937], pp. 137–8; Martianus [1983], pp. 281–2; Abbo [2003], pp. 119–20), superpartient multiples and subsuperpartient divisors (Boethius [1999], pp. 78–9; Nicomachus [1866], pp. 63–4; see also Isidorus [1911], III, 6, 12–13; Cassiodorus [1937], p. 138; Abbo [2003], pp. 120–1).

All these inequalities arise from the equality of the procedure, which Boethius demonstrates on examples of a three-member

numerical sequence. If the three same options are given, multiples arise from them first – initially doubles, then triples, quadruples, etc. Multiples are the basis for superparticular ratios that arise from reverse multiple sequences. Doubles give rise to 3 : 2 ratio, triples to 4 : 3 ratio, quadruples to 5 : 4 ratio, etc. After re-reversal of a sequence arranged in superparticular ratio a superpartient ratio arises. And even now 3 : 2 ratio is the root of 5 : 3 ratio, 4 : 3 ratio produces 7 : 4 ratio, 5 : 4 ratio establishes 9 : 5 ratio, etc. Standardly – not reversibly – arranged superparticular ratios help us to gain superparticular multiples: from 3 : 2 ratio arises 5 : 2 ratio, from 4 : 3 ratio arises 7 : 3 ratio and from 5 : 4 ratio arises 9 : 4 ratio. Similarly, according to the upwardly arranged superpartient ratios, superpartient multiples are created: 5 : 3 ratio becomes 8 : 3 ratio, 7 : 4 ratio becomes 11 : 4 ratio and 9 : 5 ratio – 14 : 5.

To understand the origin of these sequences in the given ratios we need to know three simple rules. When converting a three-member sequence (i.e. from $a - b - c$ to $a_1 - b_1 - c_1$) of one ratio to another – e.g. equality, i.e. 1 : 1 ratio, to multiples, i.e. 2 : 1 ratio – according to Boethius – and, of course, Nicomachus – the following applies (Boethius [1999], p. 81):

[P1] $a_1 = a$

[P2] $b_1 = a + b$

[P3] $c_1 = a + 2b + c$

By applying the three rules we can reconstruct the emergence of all kinds of inequalities from original identity and unity, which is the dominant framework within which, namely, Abbo of Fleury talks about these issues. Like Boethius, he claimed that every philosopher must particularly address the disciplines of the quadrivium from a philosophical point of view, because without knowledge of the four arts one cannot arrive at truth (Boethius [1999], p. 11), and that arithmetic occupies the first place among them (Boethius [1999], p. 12). Abbo also believed that philosophy, the love of wisdom, is also the love of God (Abbo [2003], p. 65, 68). The knowledge of arithmetic leads us safely to God, for God's wisdom – the symbol of which is the Temple of Wisdom built on seven pillars, corresponding to the seven liberal arts (Abbo [2003], p. 65; cf. *Prov.* 9, 1) – created everything according to its measure,

number and weight (Abbo [2003], pp. 69–71; cf. *Sap.* 11, 20). Consequently, in the beginning there could not have existed anything other than a unit, which is the principle and origin of all numbers – although it is not a number by itself: all numbers are from the unit, throughout the unit and in the unit (Abbo [2003], p. 75; cf. *Rom.* 11, 36). It also has no parts and is therefore identical with being and necessity (Abbo [2003], p. 74), and thus it is the divine foundation of the universe, which creates everything else according to mathematical ratios – so-called lambda diagram in relation to Plato's dialogue *Timaos* and Chalcidius's and Macrobius's commentaries on it (Abbo [2003], pp. 72–4; cf. Plato [1903], 34b–36d; Macrobius [1970], p. 26; Chalcidius [1975], pp. 81–2).

The author of *De arithmetica Boetii* sees the importance of the knowledge of a transfer between sequences very similarly: everything is either the Creator or the created. Everything created comes from the Creator, just as inequality is based on an original equality (Caiazzo [2000], pp. 140–1). Equality – as well as the Highest good – is identical with God and receding from this original One only inequality and evil arise (Caiazzo [2000], pp. 141–2). The knowledge of a return to equality is the knowledge of a return to God, for He is the primary Good and equality.

4. Boethius's *Introduction to Arithmetic II, 1*

However, debate over Boethius's *Arithmetic* prior to 1000 was provoked not so much by the last chapter of the first book, as by the succeeding chapter, i.e. the opening section of the second book. After explaining the birth of multiplicity in original unity, Boethius addresses the issue of returning inequality to original equality. If three numerical values are given, the organization of which corresponds to a certain ratio, then regardless of the type of inequality – multiples, superparticular or superpartient ratio, superparticular or superpartient multiple – they can be converted to equality (Boethius [1999], p. 94; Nicomachus [1866], p. 74).

Even in this case three rules are given, the obeying of which is necessary for a proper search for the return to initial equality (Boethius [1999], p. 94; Nicomachus [1866], pp. 74–5):

$$[R1] a_1 = a$$

$$[R2] b_1 = b - a$$

$$[R3] c_1 = c - (2b_1 + a)$$

Immediately after presenting these regulations, a passage follows, which was probably the decisive impulse for an individual debate of early medieval thinkers: Boethius had further said that thanks to the application of these rules we are able to convert the sequence of three numerical values ordered in a certain proportion to a more fundamental ratio. With regard to multiples, the following rule applies: quadruples are converted to triples, and triples to doubles that are then converted back to unity. Regarding superparticular ratios, 5 : 4 ratio is changed to a more authentic 4 : 3 ratio; 4 : 3 ratio subsequently gives rise to 3 : 2 ratio, from which three identical numerical values (Boethius [1999], p. 94) are derived. Boethius adds only one example to this explanation, in which quadruples gradually turn into equality – in schematic order see Fig. 1 (Boethius [1999], pp. 95–6).

1.	8 – 32 – 128	quadruples (4 : 1 ratio)	the application of rules <i>[R1–3]</i>	↓
2.	8 – 24 – 72	triples (3 : 1 ratio)	the application of rules <i>[R1–3]</i>	↓
3.	8 – 16 – 32	doubles (2 : 1 ratio)	the application of rules <i>[R1–3]</i>	↓
4.	8 – 8 – 8	equality (1 : 1 ratio)		

Fig. 1 – Boethius's example of converting multiples to equality according to [R1–3] rules

Boethius adds after this example that everyone who follows the rules presented will easily be able to find harmony (*convenientia*) in inequality, since identity is the mother of all inequalities, because whole inequality was born from it and returns back to it (Boethius [1999], p. 96).

Despite this detailed analysis of the issues, one relatively important question remained unresolved: how exactly do we transfer each type of inequality? Already Boethius denominates this part of arithmetic as the deepest (*profundissima*) science (Boethius [1999], p. 80) and the author of commentary *De arithmetica Boetii* adds that it is a mysterious and sophisticated (*obscurissima et subtilissima*) teaching (Caiazzo [2000], p. 140), yet – or because of this – there remain uncertainties, the solution of which was not considered to be apparent. If we limit ourselves here – in accordance with the main difference that emerged between

the authors of the last quarter of the 10th century – upon the mutual relationship of superparticular ratios, multiples and equality, then an alternative reading of Boethius's text is possible:

1. In accordance with *Introduction to Arithmetic* II, 1 it is possible to convert different types of superparticular ratios ($5 : 4$, $4 : 3$, $3 : 2$) and multiples ($4 : 1$, $3 : 1$, $2 : 1$) to equality ($1 : 1$) within the ranks of one type of inequality, i.e. for example from triples to doubles and then to identity, respectively from $4 : 3$ ratio to $3 : 2$ ratio and from these ratios to the same numerical values.
2. If the process of the rise of inequality from equality and the return of inequality to equality is reversible, which follows from Boethius's text, it is not possible (except for multiples) to always act exclusively within the given inequality during the search for original equality, because $3 : 2$ ratios do not have their origin in equality but in doubles, $4 : 3$ ratios in triples, etc., according to *Introduction to Arithmetic* I, 32.

In other words, if the rules from *Introduction to Arithmetic* II, 1 are followed, one cannot comply with the instructions from *Introduction to Arithmetic* I, 32 and vice versa.

5. Notker of Liège and his *De superparticularibus*

It seems that this very alternative interpretation of the two chapters from Boethius treatise was key to the whole debate on the conversion of ratio sequences at the end of the 10th century. Notker of Liège inclined towards the first option. In accordance with Boethius, he attempts to show that superparticular ratios can be converted among themselves and only during the penultimate step when $3 : 2$ ratio is converted to double, is it possible to go directly to equality. Notker, however, was well aware of the fact that the reduction of superparticular relationships between one another, without immediate conversion to multiples, put him at odds with the three rules for transitions between different ratios (i.e. rules [R1-3]) emphasised so much by Boethius. Perhaps for this reason the brief text begins with the caveat that Boethius did not intend his rule as the only possible way to transfer three unequal values arranged in a certain ratio to initial equality – it was merely an auxiliary tool that we do not need to apply

unconditionally to all cases. According to Notker, it is definitely not suitable when looking for a path from the three numerical values of a 5 : 4 relative sequence to the three identical items (Bubnov [1899], p. 297).

Notker proposes different rules for how to proceed while making individual conversions. In the case of transition from 5 : 4 to 4 : 3 ratio he recommends the following (Bubnov [1899], p. 298):

$$[N1] a_1 = a$$

$$[N2] b_1 = b - (1/2)a$$

$$[N3] c_1 = c - a$$

For transition between 4 : 3 and 3 : 2 ratio he advises observing these rules (Bubnov [1899], p. 298):

$$[O1] a_1 = a$$

$$[O2] b_1 = b - (1/2)b; \text{ resp. } b_1 = (1/2)b$$

$$[O3] c_1 = c - b$$

The search for original equality of three values within a sequence arranged in superparticular way appears, according to Notker, as follows: 5 : 4 ratio transfers to 4 : 3 ratio, then to 3 : 2 ratio which converts to double and subsequently acquires equality (Bubnov [1899], p. 298) as presented in Fig. 2.

1.	16 – 20 – 25	48 – 60 – 75	A) 5 : 4 ratio	the application of rules [N1–3]	↓
2.	16 – 12 – 9	48 – 36 – 27		the reverse of sequences	↓
3.	9 – 12 – 16	27 – 36 – 48	B) 4 : 3 ratio	the application of rules [O1–3]	↓
4.	9 – 6 – 4	27 – 18 – 12		the reverse of sequences	↓
5.	4 – 6 – 9	12 – 18 – 27	C) 3 : 2 ratio	the application of rules [R1–3]	↓
6.	4 – 2 – 1	12 – 6 – 3		the reverse of sequences	↓
7.	1 – 2 – 4	3 – 6 – 12	D) double (2 : 1)	the application of rules [R1–3]	↓
8.	1 – 1 – 1	3 – 3 – 3	E) equality (1 : 1)		

Fig. 2 – Notker's approach for conversion of 5 : 4 ratio to 1 : 1

Notker's process of transition between individual superparticular ratios does not follow the fundamental steps of Boethius's rules [R1–3] for the gradual reduction and return, but it is surely the correct approach and, moreover, it agrees with Boethius's explicit thesis that superparticular ratios can be transferred without intermediate ratios.

However, this is not how the postscript from the Tegernsee manuscript to Boethius's *Introduction to Arithmetic* ends. The author follows the way the individual inequalities are created,

and continues with three kinds of inequality, showing how these can be converted to superparticular ratios, from which it is clear how to arrive at equality.

Superpartient ratios are converted to superparticular according to the following rules:

$$[S1 = R1] \quad a_1 = a$$

$$[S2 = R2] \quad b_1 = b - a$$

$$[S3] \quad c_1 = c - \{[x + (x - y)] / y\}a; \text{ resp. } c_1 = c - [(2x - y) / y]a;$$

where x is the numerator of the given ratio (the first number) and y is the denominator (the second number) of the given ratio.⁴

Notker then moves on to the fourth and fifth kind of inequality – i.e. superparticular and superpartient multiples. Here he gives only one example, which moreover remains incomplete for superpartient multiples. However, it is relatively easy to formulate basic rules by which one should proceed in these cases:

$$[T1 = R1] \quad a_1 = a$$

$$[T2 = R2] \quad b_1 = b - a$$

$$[T3] \quad c_1 = c - [(x / y)a + b], \text{ resp. } c_1 = c - [(z - y / y)a + b],$$

where x / y is superparticular or superpartient ratio to which a particular sequence is transferred, resp. z is the numerator of a superpartient or superparticular multiple, i.e. the initial ratio.

Converting a superparticular multiple, results in a superparticular ratio, from which it is already possible to

⁴ Notker in his brief text does not give [S3] rule in the form just presented, but for each case of superpartient ratio (x / y) he refers to a specific value of the first number (a) of the given ratio that must be subtracted from the third number (c) to get the third entry of a new sequence (c_1). So, for example, for $9 : 5$ ratio it is necessary to subtract the first entry of a numerical sequence twice from a third entry and even three-fifths of the first item – i.e. $(13 / 5)a$ –, which corresponds with [S3] the rule for a ratio of $9 : 5$, then $x = 9$, $y = 5$, that is $\{[x + (x - y)] / y\}$ and $\{[9 + (9 - 5)] / 5\}a = (13 / 5)a$. In the case of $7 : 4$ ratio, according to the [S3] rule, the numbers c subtracts twice the number a and then a half of the number – i.e. $(5 / 2)a$ –, since $\{[x + (x - y)] / y\}$ and $\{[7 + (7 - 4)] / 4\}a = (10 / 4)a = (5 / 2)a$. It is same for $5 : 3$ ratio, where [S3] the rule urges us to subtract from an item c twice item a and even the third of a item – i.e. $(7 / 3)a$, because from $\{[x + (x - y)] / y\}a$ arises $\{[5 + (5 - 3)] / 3\}a = (7 / 3)a$. See Bubnov [1899], p. 298–9: in the text the value of 48 is mistakenly listed instead of the correct 49 concerning the third item of $7 : 4$ ratio.

reach a double and thus equality, as has been previously shown. When a superpartient multiple is converted, firstly a superpartient ratio arises, which it is necessary to transform to the superparticular ratio before proceeding in the same way. Using the rules [T1–3] is relatively easy. If it is necessary to reduce the multiples of two and one third (i.e. 7 : 3 ratio, *dupla sesquiertia*), then $x / y = 4 / 3$, since the rise of superparticular 4 : 3 ratio, resp. $(z - y) / y = (7 - 3) / 3 = 4 / 3$, then according to [T3] rule, the third item of a derived sequence (c_1) is given by the difference of c item and the sum of $(4 / 3)a$ with b (Bubnov [1899], p. 299).⁵ Similarly, in the case of a superpartient multiple, it is clear that, for example, in the multiple of two and three-quarters, i.e. 11 : 4 ratio (*dupla supertriparciens*), then $x / y = 7 / 4$, resp. $(z - y) / y = (11 - 4) / 4 = 7 / 4$, the third item of the numerical sequence needs to be transferred to superpartient ratio (7 : 4) by applying the [T3] rules, i.e. $c_1 = c - [(7 / 4)a + b]$ (Bubnov [1899], p. 299). By this Notker has discussed the conversions of all the inequalities to their predecessors, which ultimately lead to equality. As the only surviving manuscript of the text ends in mid-sentence, we have no choice but to reconstruct the return of individual inequalities to original equality without direct reference to Notker's *scholium*. It appears, however, that the returns of superparticular and superpartient multiples to equality might resemble what is shown in Fig. 3 and Fig. 4.

1.	A) superparticular multiple	the multiple of two and one quarter (9 : 4 ratio)	16 – 36 – 81	48 – 108 – 243	the application of rules [T1–3]	↓
2.	B) superparticular ratio	5 : 4 ratio	16 – 20 – 25	48 – 60 – 75	the application of rules [N1–3]	↓
3.			16 – 12 – 9	48 – 36 – 27	the reverse of sequences	↓
4.		4 : 3 ratio	9 – 12 – 16	27 – 36 – 48	the application of rules [O1–3]	↓
5.			9 – 6 – 4	27 – 18 – 12	the reverse of sequences	↓
6.		3 : 2 ratio	4 – 6 – 9	12 – 18 – 27	the application of rules [R1–3]	↓
7.			4 – 2 – 1	12 – 6 – 3	the reverse of sequences	↓
8.	C) multiple	double (2 : 1)	1 – 2 – 4	3 – 6 – 12	the application of rules [R1–3]	↓
9.	D) equality	equality (1 : 1)	1 – 1 – 1	3 – 3 – 3		

Fig. 3 – The conversion of superparticular multiples to equality according to Notker

⁵ The sequence of a multiple of two and one-third is given in the text with two erroneous values: The second item should be 21 instead of 15, which is given, and the correct third value is 49, not 48.

1.	A) superpartient multiple	the multiple of two and three quarters (11 : 4 ratio)	16 – 44 – 121	48 – 132 – 363	the application of rules <i>[T1–3]</i>	↓
2.	B) superpartient ratio	7 : 4 ratio	16 – 28 – 49	48 – 84 – 147	the application of rules <i>[S1–3]</i>	↓
3.			16 – 12 – 9	48 – 36 – 27	the reverse of sequences	↓
4.	C) superparticular ratio	4 : 3 ratio	9 – 12 – 16	27 – 36 – 48	the application of rules <i>[O1–3]</i>	↓
5.			9 – 6 – 4	27 – 18 – 12	the reverse of sequences	↓
6.		3 : 2 ratio	4 – 6 – 9	12 – 18 – 27	the application of rules <i>[R1–3]</i>	↓
7.			4 – 2 – 1	12 – 6 – 3	the reverse of sequences	↓
8.	D) multiple	double (2 : 1)	1 – 2 – 4	3 – 6 – 12	the application of rules <i>[R1–3]</i>	↓
9.	E) equality	equality (1 : 1)	1 – 1 – 1	3 – 3 – 3		↓

Fig. 4 – The conversion of superpartient multiples to equality according to Notker

Although it may not be obvious at first sight, Notker's rules *[S3]* and *[T3]* provide the same reduction of sequences as Boethius's rule *[R3]*, since all the rules are mutually convertible. Notker created modified rules to provide transfers to all types of inequality. However, they do not follow the wording of Boethius's rules and also, in the case of superparticular multiples, do not proceed in reverse order of the derivation of inequality, which were probably the main reasons why Abbo of Fleury and Gerbert of Aurillac objected to these instructions.

6. Gerbert of Aurillac and Abbo of Fleury

While Notker chose the first reading Boethius's *Introduction to Arithmetic*, Gerbert and Abbo chose instead the second (i.e. holding to *[R1–3]* rules) and interpreted Boethius's statement that the individual superparticular ratios should be transferred between each other differently.

Gerbert, whose treatise was already in the Middle Ages called "*Saltus Gerberti*" (Gerbert's Leap), wrote very precise and detailed instructions for the transition from superparticular 5 : 4 ratios to quadruples (4 : 1), from quadruples to triples (3 : 1), which are also means of proper transition to superparticular 4 : 3 ratios. This method of converting 5 : 4 ratio to 4 : 3 ratio, with direct reference to Boethius was introduced as correct (Bubnov [1899], pp. 32–4). 4 : 3 ratio can revert to triples (3 : 1) to provide doubles (2 : 1) in the next step and in accordance with the emergence of inequality, it applies that doubles give rise to 3 : 2 ratio, which concludes the path from 5 : 4 ratios to 3 : 2 ratios according to Boethius's rules and avoids any possible confusion (Bubnov [1899], pp. 34–5). The use of Boethius's reduction rules will readily provide equality (1 : 1) from doubles and Gerbert notes for the fourth time in his

short text that this is the only appropriate–i.e. in accordance with Boethius and, in particular, in accordance with the proper nature of numbers, numerical relationships and their origin–method of converting superparticular ratios to lower ratios of the same inequality and, if necessary, to equality itself (Bubnov [1899], p. 35). The sequence of Gerbert's steps is offered in Fig. 5.

1.	sesquiquartum	5 : 4 ratio	16 – 20 – 25	48 – 60 – 75	the application of rules [R1–3]	↓
2.	quarter	1 : 4 ratio	16 – 4 – 1	48 – 12 – 3	the reverse of sequences	↓
3.	quadruple	4 : 1 ratio	1 – 4 – 16	3 – 12 – 48	the application of rules [R1–3]	↓
4.	triple	3 : 1 ratio	1 – 3 – 9	3 – 9 – 27	the reverse of sequences	↓
5.	third	1 : 3 ratio	9 – 3 – 1	27 – 9 – 3	the application of rules [P1–3]	↓
6.	sesquitergium	4 : 3 ratio	9 – 12 – 16	27 – 36 – 48	the application of rules [R1–3]	↓
7.	third	1 : 3 ratio	9 – 3 – 1	27 – 9 – 3	the reverse of sequences	↓
8.	triple	3 : 1 ratio	1 – 3 – 9	3 – 9 – 27	the application of rules [R1–3]	↓
9.	double	2 : 1 ratio	1 – 2 – 4	3 – 6 – 12	the reverse of sequences	↓
10.	half	1 : 2 ratio	4 – 2 – 1	12 – 6 – 3	the application of rules [P1–3]	↓
11.	sesquialterum	3 : 2 ratio	4 – 6 – 9	12 – 18 – 27	the application of rules [R1–3]	↓
12.	half	1 : 2 ratio	4 – 2 – 1	12 – 6 – 3	the reverse of sequences	↓
13.	double	2 : 1 ratio	1 – 2 – 4	3 – 6 – 12	the application of rules [R1–3]	↓
14.	equality	1 : 1 ratio	1 – 1 – 1	3 – 3 – 3		

Fig. 5 – The so-called "Saltus Gerberti", i.e. the transition between superparticular ratios and multiples

Gerbert's letter seems quite clearly to be a reaction to Notker's (or a similar) reading of Boethius. Repeated refusal of immediate transfers of superparticular ratios and the repeated recommendation to always honour Boethius's rules [R1–3] or [P1–3], since only by means of their usage can we follow the process of how the numerical sequences were created, seems to be transparently targeted at a different procedure introduced in Notker's text. Gerbert did not think that Boethius, with his claims about the transfer of 5 : 4 ratio to 4 : 3 ratio, etc. meant an immediate transition from one superparticular ratio to another, but assumed intermediate stages in the form of multiple inequalities.

Abbo of Fleury actually addresses the given problem of Boethius's *Introduction to Arithmetic* in the same way as Gerbert. Directly quoting Boethius's problematic statement about mutual transfers of superparticular ratios, he immediately interprets it – Boethius meant that between 4 : 3 and 5 : 4 ratio there can never be another superparticular ratio, but we cannot take it that these ratios transfer each other without any intermediary, as superparticular ratios originate

from multiples and therefore must always be converted to multiples (Abbo [2003], p. 87, 89; cf. Boethius [1999], p. 94). Consequently, it is necessary to honour Boethius's rules *[R1–3]* (Abbo [2003], p. 86) and in accordance with the reverse order of dissimilar numerical sequences come to original equality – i.e. from superpartient multiples to superpartient ratios, from these ratios to superparticular and through multiples to equality (Abbo [2003], p. 87). Abbo subsequently presents two tables showing how the individual inequalities appeared and how we can return to perfect unity (Abbo [2003], p. 88).

Gerbert and Abbo deal with the consequences arising from *Introduction to Arithmetic* II, 1 in identical fashion and prefer to observe the emergence of inequality – according to *Introduction to Arithmetic* I, 32 – which it is necessary to revert to during a return to embryonic unity and equality, after which we proceed according to the rules *[R1–3]*. Notker's instructions for implementing the transition are possible, but they reveal some signs of confusion. In particular, they are not in accordance with the nature of numbers and the emergence of proportional sequences.

7. Importance and applications of this topic

As the previous paragraphs suggest, the theme of the introductory chapter of Boethius's *Introduction to Arithmetic* attracted considerable attention in the late 10th century. It automatically raises the question of the reason for interest in this problem – at first sight a very specific and seemingly only mathematical phenomenon. An ontological-creationist level had already been introduced and had undoubtedly formed the framework defining the whole debate i.e.: how could someone want to consider themselves a thorough believer when they did not try to reach God? If mathematical truth, numbers, numerical ratios, etc. are instruments that God used in the creation of the universe when he allowed the emergence of diversity, derived from the integral unity and identity, and if everything is transferred back in the same order to God in the end, then it is clear that not only Neo-Pythagoreans and Pythagoreans, not only Plato and Neoplatonists, but also Christians are almost obliged to explore this mysterious secret of creation and return to the Creator. But the metaphysical-

theological and anthropological context that shaped interest in a detailed analysis of geometric sequences is far from the only one to which the issue of numerical ratios – and their resulting numerical series – can be included.

Perhaps the clearest application of numerical ratios is proposed by the very definition of mathematical arts, as Boethius and Nicomachus present it in the opening passages of their *Introductions to Arithmetic*. They both mention two types of quantity (*magnitudo* and *multitudo*) both of which can only be considered in relation to itself and in relation to something else. Thus there is a division into four mathematical arts: arithmetic examines quantity as such – numbers and their characteristics); geometry is devoted to size itself – shapes and their properties; the subject of astronomy is sizes in mutual relationships – i.e. celestial bodies, their movements, etc.; and music focuses on relative amount – i.e. numerical ratios (Boethius [1999], pp. 10–1; Nicomachus [1866], pp. 5–6). Questions related to sequences, which are arranged by numerical ratios, are not covered by arithmetic, but music, according to the definition of the subjects of individual quadrivium disciplines. It is the theory of the art of music that very intensively deals with ratios, their mutual relations, connections, transfers, etc. These are topics that can be historically traced to the times of Pythagoras of Samos himself – e.g. the famous story about finding ratios in a blacksmith's workshop which Boethius informs us about (Boethius [1867], pp. 196–8).

The application of numerical ratios to music theory – e.g. chord tones, monochords, etc. – were and are fully automatic, which of course concerned early medieval authors, who often understood this issue as a natural transition between arithmetic and musical arts, as expressed, for example, in the words of Abbo of Fleury (Abbo [2003], pp. 122–3). Thematically, it is expressed also by the author of the text *De arithmetica Boetii*, when he, following Boethius, explains the identity of arithmetic and musical ratios: *diatessaron* – *sesquiertia*, 4 : 3; *diapente* – *sesquialtera*, 3 : 2; *diapason* – *dupla*, 2 : 1; and *epogdous* – *sesquioctavus*, 9 : 8 (Caiazzo [2000], p. 134, 144; cf. also Boethius [1999], pp. 200–1, [1867], pp. 201–5; Abbo [2003], pp. 73–4, 79–80). Similarly, Gerbert of Aurillac also commented on Boethius's *Introduction to music*, as mentioned above. He took an intense interest in the organ (Gerbert

[1966], p. 121; see also e.g. Sachs [1972] and used a monochord (Richerus [2000], p. 195).

However, music is not the only liberal art in which arithmetic principles of geometric sequences are actively used. It is illustrated in the initial passage of the Munich manuscript from the monastery of Tegernsee CLM 18764, f 1-1v (edited text see Evans [1978], p. 23), in which numerical ratios are used to interpret metric feet, this traditional grammatical area. All the feet are divided according to the proportion in which the stressed and unstressed sections – so-called long and short syllables – are arranged. The author, following Isidor's *Etymologiae* (Isidorus [1911], I, 17, 21–27) distinguishes (Evans [1978], p. 23):

1. an equal ratio, i.e. the numerical ratio of one foot is composed of equal parts, i.e. it represents equality – 1 : 1 ratio, such as the numbers 2 and 2 or 4 and 4;
2. a double, i.e. a ratio within one foot is such that a smaller number is twice exceeded by a larger, and hence the larger number contains in itself the smaller number twice, i.e. 2 : 1 ratio, such as the numbers 1 and 2 or 2 and 4;
3. a sesquialterum, i.e. this foot is designed in such a ratio that a larger number includes the smaller number and half of it again, because the smaller is contained in the larger one and a half times, i.e. 3 : 2 ratio;
4. a triple, i.e. such a foot in which the larger exceeds the smaller three times, i.e. 3 : 1 ratio;
5. a sesquitertium, i.e. the ratio of one foot inserts in a larger number a whole smaller number and a third of the smaller number again, i.e. 4 : 3 ratio.

In addition to this classification, there is also a list of individual metric feet: ten with the same parts – pyrrich, spondee, dactyl, anapest, proceleumaticus, dispondee, diiamb, ditrochee, antispast and choriamb; six with double portions – iamb, trochee, molossus, tribrach and major or minor ionic; only one metric foot with a triple – amphibrach; seven feet contain sesquialterum (3 : 2) multiples – cretic or amphimacer, [bacchius](#), antibacchius and first to fourth paeon; and four contain sesquitertium (4 : 3) multiple parts – from first to fourth epitrite (Evans [1978], p. 23). This classification obviously represents another significant area that actively uses numerical ratios and their relationships.

Even this list of liberal arts which actively used numerical sequences is far from complete – notably Abbo of Fleury researched in detail the application of arithmetic knowledge in other arts (especially astronomy), and he interlinked his commentary on Victorin with other fields of study in many places, (Evans and Peden [1985]).

Finally yet a different area can be mentioned – to which numerical ratios and the ability to mutually transfer them inherently belong – intellectual recreation, i.e. games. It seems most likely that it was around 1000 or just before the turn of the first millennium (thus at the same time as the arithmetic texts which this study deals with first appeared) that people in the Christian Latin West started to play the intellectual mathematical board game *rithmomachia*, i.e. the battle of the numbers. The oldest known texts about it (the so-called *Regula de rithmomachia, id est de numeri pugna* and *Ludus qui dicitur rithmimachia*) are from the first half of the 11th century (edited texts see Borst [1986], pp. 330–9, for more detailed information of the first two treatises on *rithmomachia* see Borst [1986], pp. 50–97).

If the fact that establishing the rules of the game was in the Middle Ages attributed to Gerbert of Aurillac – together with, for example, Boethius and Pythagoras (see e.g. Folkerts [2003], p. XI-5) – remains disregarded, it is also possible to think about someone close to him or his epigons (see for example Moyer [2001], pp. 20–2; Silva [2010a], pp. 146–7, [2010b], p. 224, etc.), or that manuscripts of the given era are tables on *rithmomachia*, helping tools or texts connected on a regular basis with arithmetic treatises. It is clear that the game itself is in its earliest texts always referred to as arithmetic (Borst [1986], p. 330, 335) and the issues addressed by Notker, Gerbert and Abbo serves as the necessary equipment for the player of *rithmomachia*. For this reason *rithmomachia* is often regarded as a teaching aid for understanding Boethian theoretical arithmetic. Without a detailed presentation of the rules of the game or an illustration of the use of the work with numerical ratios, it will be sufficient, if the starting position of the chips on the board is outlined. This in itself reflects Nicomachean-Boethian teaching about the relative properties of numbers and sequences (see Fig.6).

	I	II	III	IV	V-VI/VIII	6/8-5	4	3	2	1	
a	361	<u>190</u>	100					9	15	25	A
b	225	120	90					6	45	81	B
c		64	81	9			2	4	25		C
d		56	49	7			4	16	20		D
e		30	25	5			6	36	42		E
f		36	9	3			8	64	49		F
g	121	66	12					72	<u>91</u>	169	G
h	49	28	16					81	153	289	H
	I	II	III	IV	V-VI/VIII	6/8-5	4	3	2	1	

Fig. 6 – Rithmomachy, the starting position of chips

On one side of the board are chips that reflect even ratios, on the other side are the odd ratios. The values of the tokens reflect multiples, superparticular and superpartient ratios (Borst [1986], pp. 330–1, 335). Multiples of the even side are represented by a double (2 – 4, on the board plan Fig. 6 C4–C3), a quadruple (4 – 16; D4–D3), a sextuple (6 – 36; E4–E3) an octuple (8 – 64; F3–F4); while on the odd side a triple (3 – 9; fIV–III), quintuple (5 – 5; eIV–eIII), septuple (7 – 49; dIV–dIII) and nonuple (9 – 81; cIV–cIII). Superparticular conditions in an even group represent the following ratios: 3 : 2 (6 – 9, B3–A3), 5 : 4 (20 – 25, D2–C2) 7 : 6 (42 – 49; E2–F2) and 9 : 8 (72 – 81; G3–H3) and in an odd arrangement ratios such as 4 : 3 (12 – 16; gIII–hIII) 6 : 5 (30 – 36; eII–fII) 8 : 7 (56 – 64; dII–cII) and 10 : 9 (90 – 100; bIII–aIII). In both groups four superpartient ratios are also present: an even enclave includes ratios 5 : 3 (15 – 25, A2–A1), 9 : 5 (45 – 81, B2–B1), 13 : 7 (91 – 169; G2–G1) and 17 : 9 (153 – 289; H2–H1), i.e. a larger number is composed of a smaller number and of two, four, six, resp. eight parts of a smaller number; the odd ratios set includes 7 : 4 (28 – 49; hII–hI), 11 : 6 (66 – 121; gII–gI), 15 : 8 (120 – 225; bII–bI) and 19 : 10 (190 – 361; aII–aI), i.e. a larger number is composed of a small number of three, five, seven and nine parts of a smaller number (Borst [1986], p. 331).

Both sides have a special figure, the so-called Pyramid, underlined in Fig. 6, and within the game itself it is not made of one piece like other chips but of several chips put together. In the even team it resembles the value of 91 (G2), which is made up of six pieces which correspond to the identity of 1 (i.e. 1 : 1) and five-fold ratios: double (4, i.e. $2 \cdot 2$), triple (9, i.e. $3 \cdot 3$), quadruple (16, i.e. $4 \cdot 4$), quintuple (25, i.e. $5 \cdot 5$) and sextuple (36 or $6 \cdot 6$), wherein the sum of the values (1, 4, 9, 16, 25 and 36) gives 91. The odd-numbered side of the Pyramid consists of five chips and is represented by the value of 190

(aII), which is the sum of representatives of the five-fold ratios: quadruple (16, i.e. $4 \bullet 4$), quintuple (25, i.e. $5 \bullet 5$), sextuple (36, i.e. $6 \bullet 6$), septuple (49 or $7 \bullet 7$) and octuple (64, i.e. $8 \bullet 8$); for details see (Borst [1986], pp. 332–333, 337).

Even from this starting position of game pieces with numbers on the game board, it becomes clear that without a knowledge of arithmetic, especially of working with numerical ratios, it would not be possible to play the game so popular in the Middle Ages. In addition, such use of theoretical issues would fully correspond with so often declared targeting of Gerbert's practical knowledge – the so-called "Saltus Gerberti" could quite likely serve as a guide addressed to Constantine to easier understanding of the basics of this intellectual game.

8. Conclusion

The theme of numerical ratios and sequences as presented by Boethius's *Introduction to Arithmetic* engaged early medieval intellectuals so much that they dedicated several comments, commentaries or brief explanatory texts to it. It seems that in addition to theological and metaphysical reasons for interest in this issue, the multiple application of the summary of discussed theses stood in the centre of attention of late 10th century thinkers. Besides music, grammar, astronomy and other liberal arts, there was the question of conversion of inequality to equality and the search for relationships between individual numerical sets, which were arranged according to a certain type of inequality crucial for *rithmomachia*, a game popular at that time – documents date from the first half of the 11th century. The winner becomes, inter alia, someone capable of arranging the tokens with numeric values in arithmetic, geometric or harmonic sequences. This requires a knowledge of Boethius's *Introduction to Arithmetic* and thus this intellectual game was used to facilitate an understanding of the given topic.

The abundant use and, at the same time, possible differences in interpretation arising from the wording of Boethius's *Introduction to Arithmetic* seem to have become a breeding ground for a different interpretation of the conversion of inequalities to initial equality, as presented, on the one hand, by Notker of Liège and, on the other, by Gerbert of Aurillac and Abbo of Fleury.

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